High School Mathematics

Algebra II
2019-2020

Instructional Packet
Set II

Students:

Work at your own pace. Contact your teacher for support as needed.

C & I Department

Name____Class________________Date ________________

7-1 Reteaching

Exploring Exponential Models

- The general form of an exponential function is $y = ab^x$, where a is the initial amount and b is the growth or decay factor.
- To find b, use the formula $b = 1 + r$, where r is the constant rate of growth or decay. If r is a rate of growth, it will be positive. If r is a rate of decay, it will be negative. Therefore, if b is greater than 1, the function models growth. If b is between zero and 1, the function models decay. When you see words like *increase* or *appreciation*, think growth. When you see words like *decrease* or *depreciation*, think decay.
- For an exponential function, the y-intercept is always equal to the value of a.

Problem

Carl's weight at 12 yr is 82 lb. Assume that his weight increases at a rate of 16% each year. Write an exponential function to model the increase. What is his weight after 5 years?

Step 1 Find a and b.

$a = 82$ — a is the original amount.

$b = 1 + 0.16$ — b is the growth or decay factor. Since this problem models growth, r will be positive. Make sure to rewrite the rate, r, as a decimal.

$= 1.16$

Step 2 Write the exponential function.

$y = ab^x$ — Use the formula.

$y = 82(1.16)^x$ — Substitute.

Step 3 Calculate.

$y = 82(1.16)^5$ — Substitute 5 for x.

$y \approx 172.228$ — Use a calculator.

Carl will weigh about 172 lb in 5 years.

Exercises

Determine whether the function represents exponential growth or exponential decay. Then find the y-intercept.

1. $y = 8000(1.15)^x$

2. $y = 20(0.75)^x$

3. $y = 15\left(\frac{1}{2}\right)^x$

4. $f(x) = 6\left(\frac{5}{2}\right)^x$

Name____Class________________Date ________________

7-1 Reteaching (continued)
Exploring Exponential Models

You can use the general form of an exponential function to solve word problems involving growth or decay.

Problem

A motorcycle purchased for $9000 today will be worth 6% less each year. How much will the motorcycle be worth at the end of 5 years?

Step 1 Find a and b.

$a = 9000$	a is the original amount.
$b = 1 + (-0.06)$	b is the growth or decay factor. Since this problem models decay, r will be negative. Make sure to rewrite the rate, r, as a decimal.
$= 0.94$	

Step 2 Write the exponential function.

$y = ab^x$	Use the formula.
$y = 9000(0.94)^x$	Substitute.

Step 3 Calculate.

$y = 9000(0.94)^5$	Substitute 5 for x.
$y \approx 6605.13$	Use a calculator.

The motorcycle will be worth about $6605.13 after 5 years.

Exercises

Write an exponential function to model each situation. Find each amount after the specified time.

5. A tree 3 ft tall grows 8% each year. How tall will the tree be at the end of 14 yr? Round the answer to the nearest hundredth.

6. The price of a new home is $126,000. The value of the home appreciates 2% each year. How much will the home be worth in 10 yr?

7. A butterfly population is decreasing at a rate of 0.82% per year. There are currently about 100,000 butterflies in the population. How many butterflies will there be in the population in 250 years?

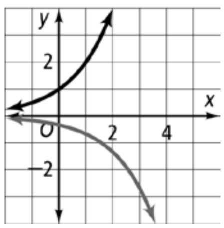
8. A car depreciates 10% each year. If you bought this car today for $5000, how much will it be worth in 7 years?

Name____Class__________________Date __________________

7-2 Reteaching
Transformations of Exponential Functions

Problem

Graph the function $g(x) = -\frac{1}{3} \cdot 2^x$ on the same set of axes as the parent function $f(x) = 2^x$. What is the effect of the transformation on the y-intercept?



First graph the parent function along with the transformed function.

$g(x) = -(\frac{1}{3})2^x$

Notice that $y = -\frac{1}{3} \cdot 2^x$ reflects the graph of the parent function $f(x) = 2^x$ across the axis and compresses it by the factor $\frac{1}{3}$.

$f(0) = 2^0 = 1$

To find the y-intercept of the parent function, substitute 0 for x in $f(x) = 2^x$. Since $f(0) = 1$, the y-intercept is (0, 1).

$g(0) = -(\frac{1}{3})2^0 = -\frac{1}{3}$

To find the y-intercept of the transformed function, substitute 0 for x in $g(x) = -\frac{1}{3} \cdot 2^x$. Since $g(0) = -\frac{1}{3}$, the y-intercept is $(0, -\frac{1}{3})$.

Exercises

Graph each function on the same set of axes as the parent function $f(x) = 2^x$. What is the effect of the transformation on the y-intercept?

1. $y = 3 - 2^x$

2. $y = -2^x$

3. $y = 2^x - 4$

4. $y = 2^x + 0.5$

5. $y = 2^{(x-5)}$

6. $y = 2^{(x+1)}$

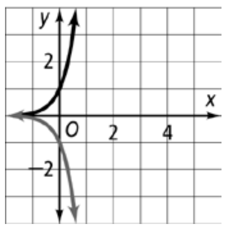
Name____Class________________Date ________________

7-2 Reteaching (continued)

Transformations of Exponential Functions

Problem

Graph the function $g(x) = -2 - 10^x$ on the same set of axes as the parent function $f(x) = 10^x$. What is the effect of the transformation on the range?



First graph the parent function along with the transformed function.

$g(x) = -2 \cdot 10^x$

Notice that $g(x) = -2 - 10^x$ reflects the graph of the parent function $f(x) = 10^x$ across the axis and stretches it by the factor 2.

$y > 0$

To find the range of the parent function, notice that for small values of *x*, the function approaches the asymptote at *y* = 0, and for large values of *x*, the function approaches infinity. Thus the range is *y* > 0.

$y < 0$

To find the range of the transformed function, notice that for small values of *x*, the function approaches the asymptote at *y* = 0, and for large values of *x*, the function approaches negative infinity. Thus the range is *y* < 0.

Exercises

Graph each function on the same set of axes as the parent function $f(x) = 10^x$. What is the effect of the transformation on the range?

7. $y = \frac{1}{2} \cdot 10^x$

8. $y = -10^x$

9. $y = 10^x - 3$

10. $y = 10^x + 1$

11. $y = 10^{(x-6)}$

12. $y = 10^{(x+5)}$

Name____Class________________Date ________________

7-5 Reteaching

Logarithmic Functions as Inverses

A logarithmic function is the inverse of an exponential function.

To evaluate logarithmic expressions, use the fact that $x = \log_b y$ is the same as $y = b^x$. Keep in mind that $x = \log y$ is another way of writing $x = \log_{10} y$.

Problem

What is the logarithmic form of $6^3 = 216$?

Step 1 Determine which equation to use.

The equation is in the form $b^x = y$.

Step 2 Find x, y, and b.

$b = 6$, $x = 3$, and $y = 216$

Step 3 Because $y = b^x$ is the same as $x = \log_b y$, rewrite the equation in logarithmic form by substituting for x, y, and b.

$3 = \log_6 216$

Exercises

Write each equation in logarithmic form.

1. $4^{-3} = \frac{1}{64}$

2. $5^{-2} = \frac{1}{25}$

3. $8^{-1} = \frac{1}{8}$

4. $11^0 = 1$

5. $6^1 = 6$

6. $6^{-3} = \frac{1}{216}$

7. $17^0 = 1$

8. $17^1 = 17$

Problem

What is the exponential form of $4 = \log_5 625$?

Step 1 Determine which equation to use.

The equation is in the form $x = \log_b y$.

Step 2 Find x, y, and b.

$x = 4$, $b = 5$, and $y = 625$

Step 3 Because $x = \log_b y$ is the same as $y = b^x$, rewrite the equation in exponential form by substituting for x, y, and b.

$625 = 5^4$

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7-5 Reteaching (continued)

Logarithmic Functions as Inverses

Exercises

Write each equation in exponential form.

9. $3 = \log_2 8$

10. $2 = \log_5 25$

11. $\log 0.1 = -1$

12. $\log 7 \approx 0.845$

13. $\log 1000 = 3$

14. $-2 = \log 0.01$

15. $\log_3 81 = 4$

16. $\log_{49} 7 = \frac{1}{2}$

17. $\log_8 \frac{1}{4} = -\frac{2}{3}$

18. $\log_2 128 = 7$

19. $\log_5 \frac{1}{625} = -4$

20. $\log_6 36 = 2$

Problem

What is the value of $\log_4 32$?

$x = \log_4 32$	Write the equation in logarithmic form $x = \log_b y$.
$32 = 4^x$	Rewrite in exponential form $y = b^x$.
$2^5 = (2^2)^x$	Rewrite each side of the equation with like bases in order to solve the equation.
$2^5 = 2^{2x}$	Simplify.
$5 = 2x$	Set the exponents equal to each other.
$x = \frac{5}{2}$	Solve for x.
$\log_4 32 = \frac{5}{2}$	

Exercises

Evaluate the logarithm.

21. $\log_2 64$

22. $\log_2 64$

23. $\log_3 3^4$

24. $\log 10$

25. $\log 0.1$

26. $\log 1$

27. $\log_8 2$

28. $\log_{32} 2$

27. $\log_9 3$

Name____Class_________________Date _________________

7-7 Reteaching

Transformations of Logarithmic Functions

The logarithmic parent function is $f(x) = \log_b x$, where b is a positive real number not equal to 1.

Vertical stretches, compressions, and reflections are applied by multiplying the function by a factor, $y = a \log_b x$.

- When $|a| > 1$, the transformation is a vertical stretch.
- When $0 < |a| < 1$, the result is a vertical compression, or a vertical shrink.
- When $a < 0$, the result is a reflection across the x-axis.

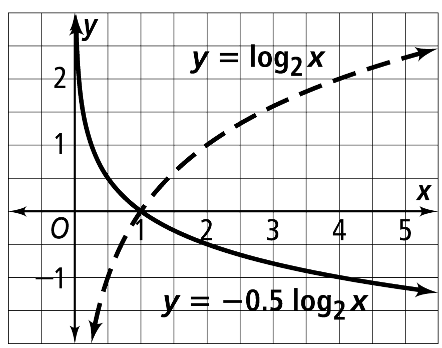
Problem

Graph the function $y = -0.5 \log_2 x$ on the same set of axes as the parent function $y = \log_2 x$. Describe the effect of the transformation on the parent function. What is the effect of the transformation on the domain?

The first step in graphing the function is to make a table of values. Include a column for the parent function and the transformed function.

Next, graph the parent function using a dashed line. Then, graph the transformed function.

x	$\log_2 x$	$-0.5 \log_2 x$
0.5	–1	0.5
1	0	0
2	1	–0.5
4	2	–1



Multiplying the parent function by –0.5 shrinks the graph vertically because $0.5 < 1$. Each y-value is closer to the x-axis for corresponding x-values. Because $-0.5 < 0$, the graph is also reflected across the x-axis. The domain remains $x > 0$.

Exercises

Graph each function on the same set of axes as the parent function $y = \log_{10} x$. Describe the effect of the transformation on the parent function. What is the effect of the transformation on the range?

1. $y = 0.25 \log_{10} x$

2. $y = -2 \log_2 x$

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7-7 Reteaching (continued)
Transformations of Logarithmic Functions

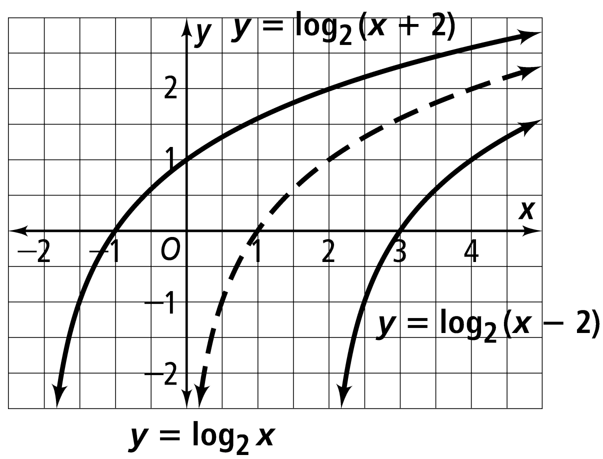
Vertical translations are applied by adding d to the logarithmic function. Horizontal translations are applied by subtracting c from x before applying the function.

- For $y = \log_b x + d$, when $d > 0$, the graph shifts up $|d|$ units
- For $y = \log_b x + d$, when $d < 0$, the graph shifts down $|d|$ units.
- For $y = \log_b (x - c)$, when $c > 0$, the graph shifts right $|c|$ units.
- For $y = \log_b (x - c)$, when $c < 0$, the graph shifts left $|c|$ units.

Problem

Sketch the graphs of $y = \log_2 x$, $y = \log_2 (x + 2)$, and $y = \log_2 (x - 2)$ on the same set of axes as the parent function. Describe how to use the parent function to graph the translations. What is the effect of the transformations on the x-intercept for each function?

Graph the parent function first.



To graph $y = \log_2 (x + 2)$, translate each point on the graph of the parent function 2 units left because $c < 0$.
The x-intercept is now (–1, 0).

To graph $y = \log_2 (x - 2)$, translate each point on the graph of the parent function 2 units right because $c > 0$.
The x-intercept is now (3, 0).

Exercises

For each given function, explain the effects of the transformations on the parent function.

3. $y = \log_{10} (x - 3)$

4. $y = \log_2 x - 3$

5. $y = \log_2 x + 4$

6. $y = \log_{10} (x + 4)$

7. $y = \log_{10} (x - 2) + 1$

8. $y = \log_2 (x + 1) - 5$

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7-8 Reteaching

Attributes and Transformations of the Natural Logarithm Function

Natural logarithmic functions are the inverse of $a = e^b$. This equation can be written as $b = \log_e a$ and also as $b = \ln a$. The term "ln" means natural logarithm.

The natural logarithm function has the domain (0, oo), the range of all real numbers, one or no x-intercept, and an asymptote of $x = 0$.

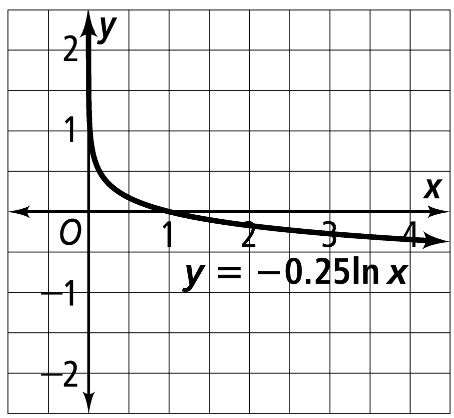
Problem

Graph the function $y = -0.25 \ln x$ and analyze the domain, range, intercepts, and asymptote.

The first step in graphing the function is to make a table of values. Use values for x that work well in the function.

x	$y = -0.25\ln x$
–1	undefined
0	undefined
$\frac{1}{e}$	0.25
e	0
e^2	–0.5

Next, sketch a graph of the function. Plot each point and connect the points.



From the graph, you can determine that the domain is (0, oo), the range is all real numbers, the x-intercept is (1, 0), and the asymptote is $x = 0$.

Exercises

Find the domain, range, x-intercept, and asymptote for each function.

1. $y = \ln x + 3$

2. $y = 4 \log_e x$

3. $y = -0.6 \ln x$

4. $y = -3 \log_e x + 4$

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7-8 Reteaching (continued)

Attributes and Transformations of the Natural Logarithm Function

Just as in other functions, transformations can be used to map the parent function to the graph of a natural logarithm function.

Natural logarithms have the same set of transformations as other logarithm bases.

$f(x) = a \log_e x$	for absolute $a > 1$, graph shows vertical stretch for $0 <$ absolute $a < 1$, graph shows vertical compression or shrink for $a < 0$, graph shows reflection across the x-axis
$f(x) = \log_e x + d$	for $d > 0$, graph shifts up d units for $d < 0$, graph shifts down d units

Problem

The time T in years at which an investment is worth x dollars is given by the function $T = (-2 \log_e x) + 10$. Describe the graph of this function as a translation, stretch, compression, or reflection of the parent function, $T = \log_e x$.

Compare the parent function $T = \log_e x$ to the transformed function using the transformation of a log function, $f(x) = a \log_e x + d$.

In this case $f(x)$ is replaced by T, so $T = a \log_e + d$.

The value of a in $T = (-2 \log_e x) + 10$ is -2. The absolute value of -2 is greater than 1, so the graph shows a vertical stretch. Because $a < 0$, the graph is also reflected across the x-axis.

The value of d in $T = (-2 \log_e x) + 10$ is 10. Because $d > 0$, the graph shows a translation of 10 units up.

Exercises

Identify the transformation that maps the first function to the second function.

5. $f(x) = \log_e x$, $g(x) = \frac{1}{3}\log_e x - 3$

6. $f(x) = \ln x$, $g(x) = -4 \ln x + 16$

7. $f(x) = \log_e x$, $g(x) = -(\log_e x) + 1$

8. $f(x) = 3 \ln x$, $g(x) = 6 \ln x - 2$

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7-6 Reteaching

Properties of Logarithms

You can write a logarithmic expression containing more than one logarithm as a single logarithm as long as the bases are equal. You can write a logarithm that contains a number raised to a power as a logarithm with the power as a coefficient. To understand the following properties, remember that logarithms are powers.

Name	Formula	Why?
Product Property	$\log_b mn = \log_b m + \log_b n$	When you multiply two powers, you add the exponents. Example: $2^6 \cdot 2^2 = 2^{(6+2)} = 2^8$
Quotient Property	$\log_b \frac{m}{n} = \log_b m + \log_b n$	When you divide two powers, you subtract the exponents. Example: $\frac{2^6}{2^2} = 2^{(6-2)} = 2^4$
Power Property	$\log_b m^n = n\log_b m$	When you raise a power to a power, you multiply the exponents. Example: $(2^6)^2 = 2^{(6 \cdot 2)} = 2^{12}$

Problem

What is $2 \log_2 6 - \log_2 9 + \frac{1}{3} \log_2 27$ written as a single logarithm?

$2 \log_2 6 - \log_2 9 + \frac{1}{3} \log_2 27 = \log_2 6^2 - \log_2 9 + \log_2 27^{\frac{1}{3}}$ — Use the Power Property twice. $6^2 = 36,\ 27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

$= \log_2 36 - \log_2 9 + \log_2 3$

$= (\log_2 36 - \log_2 9) + \log_2 3$ — Group two of the logarithms. Use order of operations.

$= \log_2 \frac{36}{9} - \log_2 3$ — Quotient Property

$= \log_2 \left(\frac{36}{9} \cdot 3\right)$ — Product Property

$= \log_2 12$ — Simplify.

As a single logarithm, $2 \log_2 6 - \log_2 9 + \frac{1}{3} \log_2 27 = \log_2 12$.

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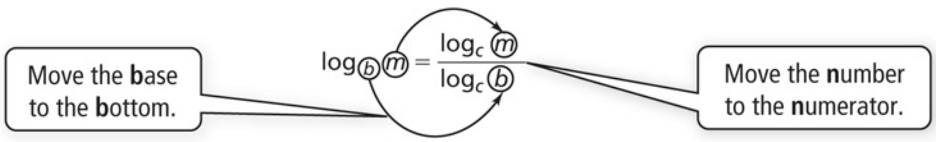
7-6 Reteaching (continued)

Properties of Logarithms

To evaluate logarithms with any base, you can rewrite the logarithm as a quotient of two logarithms with the same base.



Problem

What is $\log_4 8$ written as a quotient of two logarithms with base 2? Simplify your answer, if possible.

$\log_4 8 = \dfrac{\log_2 8}{\log_2 4}$ The base is 4 and the number is 8. Move the base to the bottom and the number to the numerator.

$= \dfrac{3}{2}$ Evaluate the logarithms in the numerator and the denominator.

Exercises

Write each logarithmic expression as a single logarithm.

1. $\log_3 13 + \log_3 3$

2. $2 \log x + \log 5$

3. $\log_4 2 - \log_4 6$

4. $3 \log_3 3 - \log_3 3$

5. $\log_5 8 + \log_5 x$

6. $\log 2 - 2 \log x$

7. $\log_2 x + \log_2 y$

8. $3 \log_7 x - 5 \log_7 y$

9. $4 \log x + 3 \log x$

10. $\log_5 x + 3 \log_5 y$

11. $3 \log_2 x - \log_2 y$

12. $\log_2 16 - \log_2 8$

Write each logarithm as a quotient of two common logarithms. Simplify your answer, if possible. (*Hint:* Common logarithms are logarithms with base 10.)

13. $\log_4 12$

14. $\log_2 1000$

15. $\log_5 16$

16. $\log_{11} 205$

17. $\log_9 32$

18. $\log_{100} 51$

Name____Class________________Date ________________

7-9 Reteaching

Exponential and Logarithmic Equations

Use logarithms to solve exponential equations.

Problem

What is the solution of $7 - 5^{2x-1} = 4$?

$7 - 5^{2x-1} = 4$	
$-5^{2x-1} = -3$	First isolate the term that has the variable in the exponent. Begin by subtracting 7 from each side.
$5^{2x-1} = 3$	Multiply each side by –1.
$\log_5 5^{2x-1} = \log_5 3$	Because the variable is in the exponent, use logarithms. Take $\log_5$ of each side because 5 is the base of the exponent.
$(2x-1)\log_5 5 = \log_5 3$	Use the Power Property of Logarithms.
$2x - 1 = \log_5 3$	Simplify. (Recall that $\log_b b = 1$.)
$2x - 1 = \frac{\log 3}{\log 5}$	Apply the Change of Base Formula.
$2x = \frac{\log 3}{\log 5} + 1$	Add 1 to each side.
$x = \frac{1}{2}\left(\frac{\log 3}{\log 5} + 1\right)$	Divide each side by 2.
$x \approx 0.84$	Use a calculator to find a decimal approximation.

Exercises

Solve each equation. Round the answer to the nearest hundredth.

1. $2^x = 5$

2. $10^{2x} = 8$

3. $5^{x+1} = 25$

4. $2^{x+3} = 9$

5. $3^{2x-3} = 7$

6. $4^x - 5 = 3$

7. $5 + 2^{x+6} = 9$

8. $4^{3x} + 2 = 3$

9. $1 - 3^{2x} = -5$

10. $2^{3x} - 2 = 13$

11. $5^{2x+7} - 1 = 8$

12. $7 - 2^{x+7} = 5$

Name____Class_______________Date _______________

7-9 Reteaching (continued)

Exponential and Logarithmic Equations

Use exponents to solve logarithmic equations.

Problem

What is the solution of 8 – log (4x – 3) = 4?

$8 - 2\log(4x - 3) = 4$	
$-\log(4x - 3) = -4$	First isolate the term that has the variable in the logarithm. Begin by subtracting 8 from each side.
$\log(4x - 3) = 4$	Multiply each side by –1.
$4x - 3 = 10^4$	Write in exponential form.
$4x - 3 = 10{,}000$	Simplify.
$4x = 10{,}003$	Add 3 to each side.
$x = \frac{10{,}003}{4}$	Solve for *x*.
$x = 2500.75$	Divide.

Exercises

Solve each equation. Round the answer to the nearest thousandth.

13. $\log x = 2$

14. $\log 3x = 3$

15. $\log 2x + 2 = 6$

16. $5 + \log(2x + 1) = 6$

17. $\log 5x + 62 = 62$

18. $6 - \log \frac{1}{2}x = 3$

19. $\log(4x - 3) + 6 = 4$

20. $\frac{2}{3}\log 5x = 2$

21. $2\log 250x - 6 = 4$

22. $5 - 2\log x = \frac{1}{2}$

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7-10 Reteaching
Natural Logarithms

The **natural logarithmic function** is a logarithm with base e, an irrational number.

You can write the natural logarithmic function as $y = \log_e x$, but you usually write it as $y = \ln x$.

$y = e^x$ and $y = \ln x$ are inverses, so if $y = e^x$, then $x = \ln y$.

To solve a natural logarithm equation:

- If the term containing the variable is an exponential expression, rewrite the equation in logarithmic form.
- If term containing the variable is a logarithmic expression, rewrite the equation in exponential form.

Problem

What is the solution of $4e^{2x} - 2 = 3$?

Step 1 Isolate the term containing the variable on one side of the equation.

$$4e^{2x} - 2 = 3$$

$$4e^{2x} = 5 \qquad \text{Add 2 to each side of the equation.}$$

$$e^{2x} = \frac{5}{4} \qquad \text{Divide each side of the equation by 4.}$$

Step 2 Take the natural logarithm of each side of the equation.

$$\ln(e^{2x}) = \ln\left(\frac{5}{4}\right)$$

$$2x = \ln\left(\frac{5}{4}\right) \qquad \text{Definition of natural logarithm}$$

Step 3 Solve for the variable.

$$x = \frac{\ln\left(\frac{5}{4}\right)}{2} \qquad \text{Divide each side of the equation by 2.}$$

$$x \approx 0.112 \qquad \text{Use a calculator.}$$

Step 4 Check the solution.

$$4e^{2(0.112)} - 2 \;\square\; 3$$

$$4e^{0.224} - 2 \;\square\; 3$$

$$3.004 \approx 3$$

The solution is $x \approx 0.112$.

Name____Class________________Date ________________

7-10 Reteaching (continued)

Natural Logarithms

Problem

What is the solution of $\ln(t-2)^2 + 1 = 6$? Round your answer to the nearest thousandth.

Step 1 Isolate the term containing the variable on one side of the equation.

$\ln(t-2)^2 + 1 = 6$

$\ln(t-2)^2 = 5$ Subtract 1 from each side of the equation.

Step 2 Raise each side of the equation to the base e.

$e^{\ln(t-2)^2} = e^5$ Definition of natural logarithm

$(t-2)^2 = e^5$

Step 3 Solve for the variable.

$t - 2 = \pm e^{\frac{5}{2}}$ Take the square root of each side of the equation.

$t = 2 \pm e^{\frac{5}{2}}$ Add 2 to each side of the equation.

$t \approx 14.182 \text{ or } -10.182$ Use a calculator.

Step 4 Check the solution.

$\ln(14.182-2)^2 \stackrel{?}{=} 5$ $\ln(-10.182-2)^2 \stackrel{?}{=} 5$

$4.9999 \approx 5$ $4.9999 \approx 5$

The solutions are $t \approx 14.182$ and -10.182.

Exercises

Use natural logarithms to solve each equation. Round your answer to the nearest thousandth. Check your answers.

1. $2e^x = 4$

2. $e^{4x} = 25$

3. $e^x = 72$

4. $e^{3x} = 124$

5. $12e^{3x-2} = 8$

6. $\frac{1}{2}e^{6x} = 5$

Solve each equation. Round your answer to the nearest thousandth. Check your answers.

7. $\ln(x-3) = 2$

8. $\ln 2t = 4$

9. $1 + \ln x^2 = 2$

10. $\ln(2x-5) = 3$

11. $\frac{1}{3}\ln 2t = 1$

12. $\ln(t-4)^2 + 2 = 5$